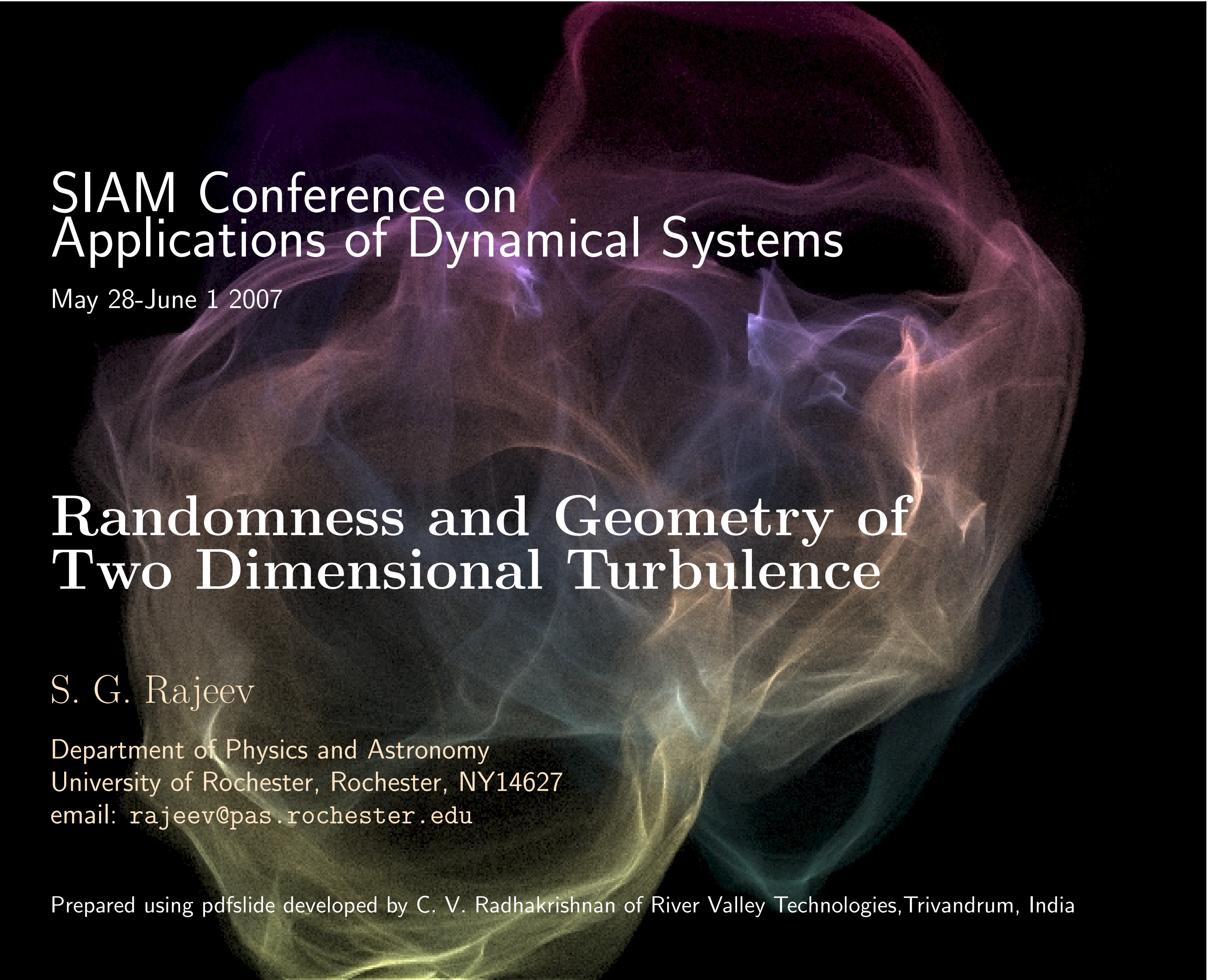

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Randomness and Geometry of Two Dimensional Turbulence

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Prepared using pdfslide developed by C. V. Radhakrishnan of River Valley Technologies,Trivandrum, India

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Coarse Graining of Fluids

See http://www.sgrajeev.com/MacroFluids for more details.

Fluid Mechanics is derived from Newtonian Mechanics by coarse graining over the motion of molecules: mathematically we pass from a large system of ODEs to a small system of PDEs. Although this is a simplification, it still leads to intractable phenomena such as turbulence. Another coarse graining, separating the small scale fluctuations from the large is needed. This can lead to new effective theories of hydrodynamics in which there is a fundamental limit to the smallest allowed distance in space: a scale much large than molecular sizes but much smaller than the overall size of the system. This is a new kind of 'fluid matter' whose basic constituents are themselves large aggregates of molecules.

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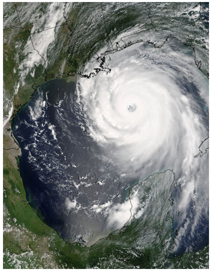
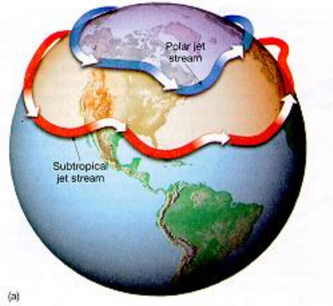
Stable Vortices in the Atmosphere





The Giant Red Spot has existed at least for four hundred years. Hurricane Katrina lasted for weeks. The jet stream is the boundary betwen two air masses at different temperatures.

The Unreasonable Stability of Large Vortices

Hydrodynamics is notoriously unstable: a butterfly flapping its wings in Australia can supposedly cause a thunderstorm in New Jersey. But then why are such large structures as hurricanes and the Gulf Stream stable? There must be an effective theory of the large scale degrees of freedom that has structures stable in spite of fluctuations. This effective theory must also have a geometrical interpretation, but must have a built in coarse-graining of space: distances cannot be allowed to be smaller than some cut-off. This must build on phenemenological models of the atmosphere and oceans that are qualitatively successful.

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An Effective Two Dimensional Theory of the Atmosphere

Consider an incompressible fluid (the atmosphere or the ocean) on the surface of a rotating sphere of radius R (the Earth) and with thickness a. Assume $a << R$.

Over distances $< a$ one cannot neglect the vertical motion of the fluid. However for large rotation(and if small scale fluctuations $\sim a$ are ignored, the fluid moves on horizontal layers. (Theorem in the book by Chemin et al). Physically reasonable: horizontal speeds in the atmosphere can be ~ 100 kmph, vertical speeds are $\sim < 10$ kmph except in localized regions (e.g., thunderstorm cells).

A first approximation for the large scale equation of motion is just Navier-Stokes in two dimensions with Coriolis force.

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Fuzzy Hydrodynamics

A better effective theory must include the ‘fuzziness’ of space caused by the coarse graining. We suggest that non-commutative geometry provides a natural framework for such a ‘fuzzy’ generalization of hydrodynamics; or we can use an analogy with quantum mechanics to get the same theory.

We will try to find such an effective theory based on geometric principles, analogous to the way the nonlinear sigma model (Landau-Ginzburg theory) of pions is found from Quantum Chromodynamics. A true derivation from first principles is too hard in both cases.

It should be possible to get a good description of the atmosphere as such an effectively two dimensional theory of the overall motion coupled to fully three dimensional theory of local cells of size a^3.

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The Euler Equations

The Euler equations of an incompressible inviscid fluid are

$$\frac{\partial}{\partial t} v + (v \cdot \nabla) v = -\nabla p, \ \operatorname{div} v = 0.$$

We can eliminate pressure p by taking a curl. Defining the vorticity $\omega = \operatorname{curl} v$ and using $(v \cdot \nabla) v = \omega \times v + \frac{1}{2} \nabla v^2$,

$$\operatorname{curl} (\omega \times v) = v \cdot \nabla \omega - \omega \cdot \nabla v \equiv [v, \omega]$$

we get,

$$\frac{\partial}{\partial t} \omega + [v, \omega] = 0.$$

Some References

A recent review,closest to our point of view : B. Khesin *Topological fluid dynamics*. Notices of the AMS, 52:1 (2005), 9-19

Similar textbook: V. I. Arnold and B. Khesin *Topological Methods in Hydrodynamics*.

For basic knowledge of fluid mechanics: L. D. Landau and Lifshitz *Fluid Mechanics* Pergamon Press.

Geophysics: Rick Salmon *Lectures on Geophysical Fluid Dynamics*

J. Y. Chemin, B. Desjardins,I. Gallagher and E. Grenier *Mathematical Geophysics* Oxford (2006)

Mathematical: S. B. Kuskin *Randomly Forced NonLinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions* European Math Soc. (2006). this is the origin of the modern theory of elliptic curves.

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More References

The idea of approximating area preserving maps by matrices originates with J. Goldstone and his student J. Hoppe in the latter's Ph. D. Thesis (MIT 1983). See also

J. Hoppe Int. J. Mod. Phys. A4 5239 (1989);

There are many applications to hydrodynamics. As a sample,

R. J. MacLachlan Phys. Rev. Lett. **71** 3043 (1993)

Rouhi, A., and H. D. I. Abarbanel Phys. Rev. E 48 (1993), 3643–3655

R. V. Abramov and A. J. Majda Proc. Nat. Sci. Acad USA bf 100 3841 (2003)

V. Zeitlin Phys Rev Lett **93** 264501(2004)

The new conserved quantity χ for two dim Euler equations was discovered in Iyer,S. V. and Rajeev, S. G. Mod.Phys.Lett.A 17 1539 (2002)

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The Poisson Bracket

In two dimensions, there are special tricks . An incompressible vector field with $\operatorname{div} v = 0$ is of the form

$$v_1 = \partial_2 \gamma, v_2 = -\partial_1 \gamma$$

for some function γ (velocity potential or stream function). Vorticity is a scalar $\omega = \Delta \gamma$. Also $\omega \cdot \nabla v = 0$ and

$$[v, \omega] = v \cdot \nabla \omega = \partial_2 \gamma \partial_1 \omega - \partial_1 \gamma \partial_2 \omega := \{\gamma, \omega\}$$

in terms of the Poisson bracket. Moreover the commutator of vector fields corresponds to the Poisson brackets of their stream functions.

$$u_a = \epsilon_{ab} \partial_b \phi, \quad v_a = \epsilon_{ab} \partial_b \gamma, \Rightarrow [u, v]_a = \epsilon_{ab} \partial_b \{\phi, \gamma\}.$$

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Hamiltonian Dynamics of Tracers

The tracer particles in a two dimensional incompressible fluid follow a hamiltonian mechanics in which the two position co-ordinates are conjugate to each other. The stream function is their hamiltonian.

$$\frac{dx^1}{dt} = \frac{\partial\gamma}{\partial x^2}, \qquad \frac{dx^2}{dt} = -\frac{\partial\gamma}{\partial x^1}.$$

This point of view leads to some nice results on tracer particles using KAM theory, as was shown by H. L. Swinnet in his Moser lecture.

Thus we are natural led to a Poisson bracket in which the two position co-ordinates are conjugate to each other: $\{x^2, x^1\} = 1.$

Two-Dimensional Euler Equations

The Euler equations can now be written as

$$\frac{\partial \omega}{\partial t} + \{\gamma, \omega\} = 0, \quad \Delta\gamma = \omega.$$

This looks just like the Euler equations of the rigid body, with the Poisson bracket replacing the commutator and the laplacian Δ replacing the moment of inertia. Thus these are geodesic equations on the group of area preserving diffeomorphisms: the laplacian Δ determines the metric tensor. But the curvature here is negative in most planes, leading to instabilities of the fluid motion.

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Conservation Laws

In the absence of dissipation $Q_f = \int f(\omega(x))dx$ is conserved for all functions $f : R \to R$. This is obvious from

$$\frac{dQ_f}{dt} = \int f'(\omega)\{\omega, \gamma\}dx = \int \{f(\omega), \gamma\}dx = 0.$$

The energy is

$$\frac{d}{dt}\int \gamma\omega dx = 2\int \gamma\{\omega, \gamma\}dx = \int \{\omega, \gamma^2\}dx = 0.$$

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A New Conserved Quantity

A conserved quantity for two dimensional Euler equations that seems not to have been noticed before is

$$\chi = -\mathcal{P}\int \log|\omega(x) - \omega(y)|dxdy.$$

To see this, define the distribution

$$\rho(\lambda) = \int \delta(\omega(x) - \lambda)dx, \quad f(\omega) = \int f(\lambda)\rho(\lambda)d\lambda.$$

Now

$$\chi = -\mathcal{P}\int \log|\lambda - \lambda'|\rho(\lambda)\rho(\lambda')d\lambda d\lambda'.$$

Please make this physicist's argument more rigorous! Related to the 'free entropy' of probability distributions on the real line.

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Viscosity

If we start with Navier-Stokes instead of Euler,

$$\frac{\partial\omega}{\partial t}+\{\gamma,\omega\}=-\nu\Delta\omega$$

where ν is the inverse Reynolds number. The energy is no longer conserved:

$$\frac{d}{dt}\left[\frac{1}{2}\int\gamma\omega dx\right]=-\nu\int\omega^2dx<0.$$

For convex functions

$$\frac{d}{dt}\int f(\omega)dx=-\nu\int f''(\omega)|\nabla\omega|^2dx\leq 0.$$

Effect of Viscosity on χ

$$\frac{d}{dt}\mathcal{P}\int \log|\omega(x)-\omega(y)|dxdy = \nu\int \frac{\nabla^2\omega(x)-\nabla^2\omega(y)}{\omega(x)-\omega(y)}dxdy.$$

The r.h.s is negative for eigenfunctions of the Laplace operator. Is it always negative?

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An Uncertainty Principle for Macroscopic Fluid Mechanics

If we are interested in deriving an effective theory for large objects, we must average the Navier-Stokes equations over some area a^2 characteristic of these objects. It shouldn't make sense to localise a tracer particle in this fluid over a distance shorter than a, given only these averages. This is reminiscent of the uncertainty principle of quantum mechanics, with a^2 playing the role of Plank's constant. Tracer partucles in macroscopic fluid mechanics have co-ordinates which are operators satisfying

$$[x^1, x^2] = -ia^2 \Rightarrow \Delta x^2 \Delta x^1 \geq a^2.$$

This fits naturally with the Poisson brackets for tracer particles we described earlier, $\{x^2, x^1\} = 1$.

Principles for MacroFluid Mechanics

Just as the passage from molecular to fluid description preserved all the conserved quantities, the passage to macroscopic fluid dynamics should satisfy certain principles:

1. The conserved quantities are preserved in the case without dissipation. When there is dissipation these quantities decay at the appropriate rate.
2. Geometric character of the system as a hamiltonian plus gradient flow is preserved.
3. The Lie algebra of incompressible vector fields must become a finite dimensional Lie algebra.

Quantum Mechanics and Coarse-Graining

If we were to quantize a system whose phase space is the sphere, each real function on the sphere would be represented by a hermitean matrix. The area of the two sphere being finite, the dimension of this matrix will be finite , the number of linearly independent quantum states ($N \sim \frac{\text{Area}}{\hbar}$). The Poisson bracket of functions will go over to commutator of matrices.

Clearly we will lose some information in this passage to quantum mechanics: only N^2 real numbers are needed to give a hermitean $N \times N$ matrix. In fact it is possible to think of the matrix $\hat{f}$ associated to each function as an average over a region of area $\hbar$ in phase space.

Coherent States

We can think of the passage from the function ω to the matrix $\hat{\omega}$ as averaging with respect to certain matrix-valued functions which are essentially the projection operators to the coherent states of the quantum system:

$$\hat{\omega} = \int \hat{P}(x)\omega(x)dx, \quad \omega(x) = \operatorname{tr}\hat{\omega}\hat{P}(x)$$

For a sphere there is an explicit formula in terms of rotation matrices $R(g)$ in spin j, $\hat{P}(x) = R(g)pR(g)^{-1}, \sigma{\cdot}x = g\sigma_3 g^{-1}, p = \mathrm{diag}(1, 0, \cdots 0)$. By applying the average to Poisson's equation, we can also calculate the matrix elements of $\hat{\Delta}$, the averaged version of the Laplace operator which is linear operator in the space of hermitean matrices.

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Poisson Brackets and Commutators

Standard arguments of quantum mechanics show that for large N, the commutator of operators approaches the Poisson brackets of functions:

$$[\hat{a},\hat{b}] = \frac{i}{N}\widehat{\{a,b\}} + \mathrm{O}(\frac{1}{N^2}).$$

The Lie algebra of Poisson brackets of functions becomes the finite dimensional Lie algebra $SU(N)$ of hermitean matrices. Similarly

$$\mathrm{tr}\hat{a} = \int a dx$$

Matrix-valued Averages in Hydrodynamics

Imagine that the vorticity $\omega$is given as a function on the sphere of unit area. If we average it over regions of area a^2 we will get $N = \frac{1}{a^2}$ real numbers. What we argue is that these be arranged into a hermitean matrix:

$$\hat{\omega} = \int \omega(x)\hat{P}(x)dx.$$

The matrix $\hat{P}(x)$ drops rapidly (like a Gaussian) to zero outside of a region of area a^2 surrouding x. Thus these are some fuzzy or smooth averages of the original function ω.

Macroscopic Hydrodynamic Equations

By applying this averaging procedure to the Navier Stokes equation we will get our equations for macroscopic fluid mechanics. Small scale effects can lead to a noise term. The covariance of the noise is determined by the fluctuation-dissipation 'theorem': that the energy lost to dissipation must be balanced by that gained by fluctuations.

$$\frac{\partial\hat{\omega}}{\partial t}+i[\hat{\gamma},\hat{\omega}]=\nu\hat{\Delta}\omega+\eta,\quad \hat{\Delta}^{ad}_{bc}\gamma^{c}_{d}=\omega^{a}_{b}-2\Omega\hat{n}_3,\quad <\eta^{a}_{b}(t)\eta^{c}_{d}(t')>=\delta(t-t')G^{ab}_{cd},\quad G^{ab}_{cd}=\tau\Delta^{ae}_{cf}\Delta^{fb}_{ed}.$$

Just for fun we also added a Coriolis term, when the sphere is rotating. Much simpler, because these are Stochastic ODE. Without noise this was done by V. Zeitlin Phys Rev Lett **93** 264501(2004) by slightly different arguments.

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Conservation Laws

We preserve the conservation laws in the absence of dissipation and fluctuations. For example

$$Q_f = \operatorname{tr} f(\hat{\omega}), \quad H = \frac{1}{2}\operatorname{tr}\hat{\gamma}\hat{\omega}$$

are conserved. Also

$$\chi = \sum_{i \neq j} \log|\lambda_i - \lambda_j|$$

is conserved generalizing the new conserved quantity we found earlier.

Numerical Solution

This can be solved numerically on a PC for small values of the dimension. Alternatively, we can study the corresponding Fokker-Plank equation mathematically and see that the solution tends to a Gibbs distribution dominated by stable vertices . We can see numerically that even with a zero initial condition, stable vertices will form rather quickly. Their boundaries fluctuate but vortices last a long time until they are killed off by dissipation. This would take a supercomputer to see if we were to directly solve Navier-Stokes .

Fokker-Plank Equation

A system of stochastic ODE is equivalent to a linear PDE:

$$\frac{\partial W}{\partial t} = G^{ab}_{cd}\frac{\partial^2 W}{\partial\hat{\omega}_{ab}\partial\hat{\omega}_{cd}} + \nu\Delta^{ab}_{cd}\omega_{ab} + \gamma^{ab}L_{ab}W$$

where $L_{ab} = -i\left(\omega_{ac}\frac{\partial}{\partial\omega_{cd}} - \omega_{bc}\frac{\partial}{\partial\omega_{ca}}\right)$ satisfy the commutation relations of $U(N)$. The asymptotic behavior of these is an equilibrium (Gibbs state) dominated by large vortices. Please help with proving this rigorously.

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Mesoscopic Fluid Dynamics

We have on the one hand a PDE for the real variable ω and on the other hand an ODE for the matrix $\hat{\omega}$. Imagine an intermediate situation where the matrix $\hat{\omega}(x)$ varies slowly over large distances and will satisfy some system of PDE generalizing Navier Stokes. This interpolates between the two extremes and is likely to be useful as an effective theory of such two dimensional fluids.

$$\frac{\partial\hat{\omega}(x)}{\partial t} + i\{[\hat{\gamma}(x),\hat{\omega}]\} = (\nu\hat{\Delta} + \tilde{\nu}\Delta)\hat{\omega}, \quad \hat{\Delta}^{ad}_{bc}\gamma^c_d + \Delta\gamma^a_b = \omega^a_b.$$

$$\{[\hat{\gamma}(x),\hat{\omega}]\}^a_b = \hat{\gamma}^a_c(x)\hat{\omega}^c_b(x) - \hat{\omega}^a_c(x)\hat{\gamma}^c_b(x) + \epsilon^{ij}\partial_i\gamma^a_c\partial_j\omega^c_b - \epsilon^{ij}\partial_i\omega^a_c\partial_j\gamma^c_b.$$

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Coarse-Graning of Fluids in Three Dimensions

Details in arxiv:0705.2139

Any function $f : R^3 \to C$ has the Fourier representation

$$f(x) = \int \tilde{f}(k) e^{2\pi i k\cdot x} dk.$$

The product of two functions corresponds to the convolution defined through the addition of momenta:

$$f_1 f_2(x) = \int \tilde{f}_1(k_1)\tilde{f}_2(k_2) e^{2\pi i [k_1+k_2]\cdot x} dk_1 dk_2.$$

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We can impose a cutoff on large momenta by requiring the momenta to take values in S^3 instead of R^3. We must then change the composition law for momenta to group multiplication in $SU(2)$. Then the multiplication of functions in space becomes non-commutative.

$$f(x) = \int \tilde{f}(g) e^{2\pi i k(g)\cdot x} dg, \quad f_1 f_2(x) = \int \tilde{f}_1(g_1)\tilde{f}_2(g_2) e^{2\pi i [k(g_1 g_2)]\cdot x} dg_1 dg_2.$$

Nevertheless, we found the cutoff version of Euler's equation in three dimensions. The trick is to use the Clebsch parametrization, which turns an ideal fluid into a hamiltonian system.

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Conclusions

1. Don't try to follow the complicated dynamics of unstable systems: instead look for effetive theories for collective degrees of freedom
2. Quantum geometry provides discretization methods for hydrodynamics, potentially useful for numerical solutions. Non-commutativity of the collective degrees of freedom are natural even in classical physics.
3. The Fokker-Plank equation for turbulent flows is as hard to understand fully as strongly nonlinear quantum field theories: it is a toy model for theories quantum gravity.
4. Recently found a way to extend the non-commutative methods to three dimensional hydrodynamics. Hope to go beyond purely theoretical models.

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